Dynamical Systems and Ergodic Theory PhD Exam – Spring 2012

Introduction: This is the first year for the Dynamical Systems and Ergodic Theory PhD exam. As such the material on the exam will conform fairly closely to the topics in this year's MTG6401 and MTG6402. In future years a standard syllabus will be developed and the exam and courses will cover that material. As such, the sample problems listed below probably give you the best idea of what material to expect on the Spring, 2012 exam.

Topics:

- Topological Dynamics – Definitions and basic results about the following for maps and flows.
  - periodic points and orbits
  - alpha and omega limit sets
  - recurrence, recurrent set
  - nonwandering set
  - transitivity (both forward orbit and full orbit)
  - topologically mixing
  - minimal sets
  - sensitivity on initial data
  - conjugacy and semicongugacy
  - stability of fixed and periodic points
- Symbolic Dynamics – Definitions and basic results about the following
  - cylinder sets
  - metrics
  - subshifts of finite type
  - The Perron-Fröbenius Theorem and its consequences.
- Differential Equations and flows – Definitions and basic results about the following
  - stability of rest points
  - stability of periodic orbits and Floquet multipliers

- Ergodic theory – Definitions and basic results about the following
  - von Neumann's L^2 ergodic theorem
  - Birkhoffs pointwise ergodic theorem
  - ergodicity and strong mixing
  - Bernoulli and Markov measures on subshifts of finite type.
- Area preserving dynamics – Definitions and basic results about the following
  - Poincaré Recurrence Theorem
  - Connections of ergodicity and topological dynamics
  - existence of invariant measures
  - The space of invariant measures

Book sections:

These sections give you some idea of the material. More will be added. But as noted above, for this year's exam the best guide is the course notes for MTG6401/2, the problems assigned in those courses, and the problem list below.

- Walters, P. *An Introduction to Ergodic Theory*, Springer-Verlag, 1982. Sections 1.1–1.6 and in Section 1.7, just Lemma 1.18, Theorem 1.19 and Definition 1.5(ii), Sections 5.1–5.4, Sections 6.1–6.4
- Brin, M. and Stuck, G. *Introduction to Dynamical Systems*, Cambridge, 2002. Sections 2.1–2.3, 3.1–3.3, 4.1–4.6.
- Deavaney, R., An Introduction To Chaotic Dynamical Systems, Westview Press, 2003 (or older editions). Sections 2.1–2.4.
- Katok, A. and Hasselblatt, B., Introduction to the Modern Theory of Dynamical Systems, Cambridge University Press, 1997. Sctions 1.1-1.3, 1.8, 1.9, and 2.5.

Notation:

- The statement that $(X, \mathcal{B}, \mu, f)$ is a *mpt* (measure preserving transformation) means that X is a set with the probability measure μ defined on the σ-algebra $\mathcal{B}$ and f is a bijective, bi-measurable map $f : X \to X$ which preserves the measure μ.
- The statement that $(X, d, \mathcal{B}, \mu, f)$ is a *mph* (measure preserving homeomorphism) means that $(X, \mathcal{B}, \mu, f)$ is a mpt and in addition, (X, d) is a compact metric space, f is a homeomorphism, and $\mathcal{B}$ is the Borel σ-algebra.
- The circle is $S^1 = \mathbb{R}/\mathbb{Z}$ and the two-dimensional torus is $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
- $\mathrm{supp}(\mu)$ means the support of the measure μ

- $o(x, f) = \{\dots, f^{-1}(x), x, f(x), f^2(x), \dots\}$ is the full orbit of x under the invertible map f. When f is perhaps non-invertible, the forward orbit of x is $o_+(x, f) = \{x, f(x), f^2(x), \dots\}$
- For a matrix A, its spectral radius is the magnitude of its largest eigenvalue and is denoted $\rho(A)$.

Problems:

1. For $R_\omega : S^1 \to S^1$ defined by $R_\omega(\theta) = \theta + \omega$, show that R_ω is ergodic with respect to Lebesgue measure if and only if $\omega \notin \mathbb{Q}$ (note you may assume that R_ω preserves Lebesgue measure).
2. Let (X, d) be a compact metric space and $f : X \to X$ a homeomorphism.
   (a) Define the *recurrent set* $\Lambda(f)$ and the *nonwandering set* $\Omega(f)$ and show they are both compact, invariant sets.
   (b) Show that $\Lambda(f) \subset \Omega(f)$ and give an example where the inclusion is proper.
   (c) Define the *omega-limit set* $\omega(x, f)$ of a point $x \in X$ and show it is a compact, invariant sets.
   (d) Prove or disprove: For every $x \in X$, that $\omega(x, f) \subset \Lambda(f)$.
3. Assume $(X, d, \mathcal{B}, \mu, f)$ is a mph.
   (a) Define $\mathrm{supp}(\mu)$ and show it is a compact, invariant set.
   (b) Show that almost every point in $\mathrm{supp}(\mu)$ is recurrent.
   (c) Show that every point in $\mathrm{supp}(\mu)$ is nonwandering.
   (d) Show that $\mu(\Omega(f)) = 1$.
   (e) If $(X, d, \mathcal{B}, \mu, f)$ is ergodic, show that f is transitive on $\mathrm{supp}(\mu)$.
   (f) show that $\mathrm{supp}(\mu) \subset \Lambda(f)$, where $\Lambda(f)$ is the recurrent set of f
4. Let (X, d) be a compact metric space and $f : X \to X$ a homeomorphism.
   (a) Show that there is always an f-invariant subset $Z \subset X$ so that f restricted to Z is minimal.
   (b) Show that there is always a measure μ so that $(X, d, \mathcal{B}, \mu, f)$ is a mph.
5. Define what it means for the mpt $(X, \mathcal{B}, \mu, f)$ to be ergodic and to be strong mixing. Show that mixing always implies ergodic and give an example which shows the converse is false.

6. Assume $(X, \mathcal{B}, \mu_0, f)$ and $(X, \mathcal{B}, \mu_1, f)$ are both *ergodic* mpt.
   (a) Show there exist sets X_0 and X_1 which are both f-invariant and $\mu_0(X_0) = 1$, $\mu_0(X_1) = 0$, $\mu_1(X_0) = 0$ and $\mu_1(X_1) = 1$.

   (b) If $0 < r < 1$ and $\mu_r := (1-r)\mu_0 + r\mu_1$, show that $(X, \mathcal{B}, \mu_r, f)$ is a mpt that is *not* ergodic.

7. If $f : X \to X$ is a homeomorphism of a compact metric space which is full orbit transitive and $\alpha : X \to \mathbb{R}$ is a continuous invariant function, $\alpha \circ f = \alpha$, prove that α is a constant function.

8. Assume that $f : X \to X$ is a homeomorphism of a compact metric space, prove that the following are equivalent.
   (a) For every $x \in X$, its full orbit $o(x, f)$ is a dense subset of X,

   (b) If Z is a nonempty, compact subset of X with $F(Z) = Z$, then $Z = X$.

   (c) for every nonempty open subset $U \subset X$,
$$\bigcup_{k \in \mathbb{Z}} f^k(U) = X.$$

9. Assume $(X, \mathcal{B}, \mu, f)$ is a mpt and $A \in \mathcal{B}$ is such that $f(A) \subset A$, define a set $A' \subset A$ and prove that $f(A') = A'$ and $\mu(A') = \mu(A)$.

10. Let A be an $n \times n$-matrix with all entries equal to zero or one, and A does not have a row of all zeros.
    (a) Define the one-sided subshift of finite type Σ_A^+ determined by A and the shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$.

    (b) Define one of the standard metrics on Σ_A^+.

    (c) Show that periodic points of σ are dense in Σ_A^+.

    (d) Show that (Σ_A^+, σ) is forward orbit transitive if and only if for all $1 \le i, j \le b$ there is a $k \ge 0$ so that $(A^k)_{i,j} > 0$.

    (e) Now assume there is some $m \ge 0$ so that $A^m > 0$ and for each $k > 0$ let N_k be the number of fixed points of fixed points of σ^k in Σ_A^+. Prove that
$$\lim_{k \to \infty} \frac{\log(N_k)}{k} = \log(\rho(A)),$$
    where $\rho(A)$ is the spectral radius of A.

11. Show that the mpt $(X, \mathcal{B}, \mu, f)$ is ergodic if and only if for all $A \in \mathcal{B}$ with $\mu(A) > 0$,
$$\mu(\{x \in X : \text{there exists } n \in \mathbb{Z} \text{ with} f^n(x) \in A\}) = 1.$$

12. Let A be a 2×2-matrix with integer entries and determinant one.
   (a) Describe the construction of the linear toral automorphism $\phi_A : \mathbb{T}^2 \to \mathbb{T}^2$ derived from A.

   (b) Show that the set of periodic points of ϕ_A are dense in $\mathbb{T}^2$.

   (c) Now let
$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$
   Prove that ϕ_A is full orbit transitive.

   (d) Now let
$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$
   Prove that ϕ_A is not full orbit transitive.

13. Let f be a homeomorphism of the compact metric space X. Show the following are equivalent.
   (a) f is full orbit transitive

   (b) If U is a non-empty open set with $f(U) = U$, then U is dense.

   (c) For all nonempty, open U and V there is an $n \in \mathbb{Z}$ with $f^n(U) \cap V \neq \emptyset$.

   (d) There exists a dense, G_δ set $Z \subset X$ so that $z \in Z$ implies $o(z, f)$ is dense in X.

14. The Contraction Mapping Theorem: If (X, d) is a complete metric space and $f : X \to X$ satisfies for some $0 < k < 1$ and for all $x, y \in X$
$$d(f(x), f(y)) \leq k\, d(x, y),$$
then f has a unique fixed point P, and for any point $x \in X$, $f^n(x) \to P$ as $n \to \infty$.

15. (a) Define sensitive dependence on initial data.

   (b) If (Σ_A^+, σ) is the one-sided subshift of finite type defined by the transition matrix A with $A^k > 0$ for some $k > 0$, show that (Σ_A^+, σ) has sensitive dependence on initial data everywhere.

16. Find the fixed points of this differential equation and classify their stability type
$$\begin{aligned} \dot{x} &= x^2 - y - 1 \\ \dot{y} &= y^2 - 3y(x + 1). \end{aligned}$$

17. Prove the one-dimensional Brouwer fixed point theorem: If $f : [0,1] \to [0,1]$ is continuous, then it has a fixed point.

18. If $f : S^1 \to S^1$ is an **orientation reversing** homeomorphism, show that f has at least two fixed points.

19. A homeomorphism between two metric spaces $h : (X, d_X) \to (Y, d_Y)$ is called *bi-Lipschitz* if there is a constant K so that for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,

$$d_Y(h(x_1), h(x_2)) \leq K\ d_X(x_1, x_2) \text{ and } d_X(h^{-1}(y_1), h^{-1}(y_2)) \leq K\ d_Y(y_1, y_2).$$

Prove that if $f : X \to X$ and $g : Y \to Y$ are conjugate by a bi-Lipschitz homeomorphism $h : (X, d_X) \to (Y, d_Y)$ (i.e. $h \circ f = g \circ h$) and f has sensitive dependence on initial data everywhere, then so does g.

20. If $f : \mathbb{R} \to \mathbb{R}$ is a C^2-map with $f(0) = 0$ and $|f'(0)| < 1$, prove directly (i.e. without quoting a general stability theorem) that there is a neighborhood $B_\epsilon(0)$ for some $\epsilon > 0$ so that $x \in B_\epsilon(0)$ implies that $\lim_{n\to\infty} f^n(x) = 0$.

21. Given $f : X \to X$ and $g : Y \to Y$, define what it means for f to be *semiconjugate* to g. For each of the following dynamical propertes, show that if f is semiconjugate to g and f has the property, then g does also. You may assume that X and Y are compact metric spaces. Also, in each case give an example where g has the property, but f does not.
    (a) Full orbit transitive.
    (b) Minimal.
    (c) The collection of periodic orbits is a dense set.
    (d) Every point is non-wandering.
    (e) Every point is recurrent.

22. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x + 1, y)$. Show that f does not have any invariant, Borel probability measures.

23. (a) For a map $f : \mathbb{R}^n \to \mathbb{R}^n$ with $f(0) = 0$ define what it means for the origin to be a Lyapunov stable fixed pointy, an asymptotcially stable fixed point, and an unstable fixed point.
    (b) Let A be a (2×2)-matrix and define $f : \mathbb{R}^2 \to \mathbb{R}^2$ as

    $$f(x, y) = A \begin{pmatrix} x \\ y \end{pmatrix}$$

    Show that the origin is asymptotcally stable if and only of $\rho(A) < 1$. You must prove this directly and not just quote a stability theorem.

24. Assume $0 < p < 1$.
   (a) Define the Bernoulli measure μ_p on the two-sided shift on two symbols Σ_2.

   (b) Show that $(\Sigma_2, \mathcal{B}, \sigma, \mu_p)$ is mixing (you don't have to prove it is a mpt, you may assume that is true).

   (c) Define $\alpha : \Sigma_2 \to \mathbb{R}$ by
   $$\alpha(\ldots s_{-1}s_0s_1s_2) = -s_0 + 2s_1 + s_2^2.$$
   Compute the following limit for μ_p-almost every sequence in $\underline{s} \in \Sigma_2$:
   $$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \alpha(\sigma^i(\underline{s})).$$

25. A continuous function of a compact metric space $f : X \to X$ is called *locally eventually onto (leo)* if for any open set U there is a $k \geq 0$ with $f^k(U) = X$.
   (a) Show that if f is leo it is forward orbit transitive.

   (b) Show that if f is leo it has sensitive dependence on initial data everywhere.

   (c) Show that the full tent map T is leo where $T : [0,1] \to [0,1]$ is defined by $T(x) = 2x$ for $0 \leq x \leq 1/2$ and $T(x) = -2x - 2$ for $1/2 \leq x \leq 1$.

26. Define $f : S^1 \to S^1$ by $f(\theta) = 3\theta$. Show that
   (a) f is forward orbit transitive.

   (b) f has sensitive dependence on initial conditions everywhere

   (c) The collection of periodic orbits of f are dense in S^1.

   (d) Every point is nonwandering under f.

   (e) Let $Z = \{x \in S^1 :$ there exists $n \geq 0$ with $f^n(x) = 0\}$. Show Z is dense in S^1.

27. Assume that X is compact metric and is connected.
   (a) Give the definition of a *flow* ϕ_t on X.

   (b) Define the omega limit set $\omega(x, \phi_t)$ for $x \in X$.

   (c) Show that for all x, $\omega(x, \phi_t)$ is connected.

28. Let $f : X \to X$ be a continuous function where X is a compact metric space. Show that for al $x \in X$,
   $$\omega(x, f) = \bigcap_{n\in\mathbb{N}} Cl\{f^k(x) : k \geq n\}.$$

29. Suppose that $f : X \to X$ is continuous, where X is a compact metric space. Suppose that for some point $x \in X$, $\omega(x) = \{p, q\}$ where p and q are distinct points of X. Prove that $f(p) = q$ and $f(q) = p$.

30. Suppose that X is a compact metric space. Suppose that $f : X \to X$ is a homeomorphism and $d(f(x), f(y)) = d(x, y)$ for each pair of points x and y is X.
    (a) Prove that each point $x \in X$ is recurrent.

    (b) Prove that each point $x \in X$ is almost periodic.

31. Find all homeomorphisms of the interval $[0, 1]$ which preserve Lebesgue measure. Are any of these ergodic? Prove your answer.

32. Let $(X, \mathcal{B}, \mu, f)$be a mpt. Let N be a positive integer, and let $A \in \mathcal{B}$ have $\mu(A) > 0$. Prove that there there exists an integer $n > N$ with $\mu(f^{-n}(A) \cap A) > 0$.

33. Let $(X, \mathcal{B}, \mu)$ be the particular probability space where X is the circle in the plane with center at the origin and radius 1, $\mathcal{B}$ is the collection of Borel sets, and μ is Lebesgue measure. Let $T : X \to X$ denote rotation by $\frac{2\pi}{3}$ radians. Recall that T is measure preserving. Let B denote the set of points $(x, y) \in X$ with $x > 0$ and $y > 0$, and let f denote the characteristic function of B. Give an explicit description of the function f^* in the Birkhoff Ergodic Theorem.

Problems for Spring 2013 Exam:

1. Let $f : X \to X$ be a continuous map of a compact metric space to itself. Suppose that for any points $x, y \in X$ we have $d(f(x), f(y)) \leq d(x, y)$. Prove that the topological entropy of f is zero.

2. Let $f : X \to X$ be a continuous map of a compact metric space to itself. Suppose that there exist disjoint closed subsets, V and W of X such that $f(V) = X = f(W)$. Prove that the topological entropy of f is at least $\log(2)$.

3. Let X denote the Morse Minimal Set, and let $f : X \to X$ denote the restriction of the full one-sided shift on two symbols to X. We know that X is a minimal set for f. Is X also a minimal set for f^2? Prove your answer.