University of Florida, Department of Mathematics

Algebra First-Year Practice Exam B, Part 2

Answer four problems. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. This problem has three parts.

(a) Prove that a prime $p \in \mathbb{N}$ such that $p \equiv 3 \pmod 4$ is prime when considered as an element in the ring of Gaussian integers $\mathbb{Z}[i]$.

(b) Prove that a prime $p \in \mathbb{N}$ such that $p \equiv 1 \pmod 4$ is a product of two non-associate primes in $\mathbb{Z}[i]$.

(c) Give a factorization of 30 as the product of a unit and powers of non-associate primes in $\mathbb{Z}[i]$.

2. Let R be the ring of $n \times n$ matrices over a field F, where $n \geq 2$.

(a) Show that R has no ideals other than 0 and R.

(b) Exhibit a nonzero proper left ideal of R.

3. Find one representative of each conjugacy class of elements of order 4 in the group $GL(6, \mathbb{Q})$.

4. Let F be a field.

(a) What does it mean to say that a field extension of F is algebraic?

(b) Let $F \subset K \subset L$ be three fields. Prove that if K is algebraic over F and L is algebraic over K, then L is algebraic over F.

5. A commutative ring R with 1 is called a local ring if and only if it has a unique maximal ideal. Suppose that R is a ring with a proper ideal M. Show that R is local, with maximal ideal M, if and only if every element of $R \setminus M$ (set difference) is a unit.