University of Florida, Department of Mathematics

Algebra First-Year Practice Exam A, Part 1

Answer four problems. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G be a group of order 44. Prove that

(a) a Sylow 11-subgroup of G is normal in G, and

(b) the centre of G contains an element of order 2.

2. Let $G = A \times B$ be a finite group which is the (internal) direct product of two subgroups A and B which are nonabelian simple groups.

(a) Show that the only proper, nontrivial normal subgroups of G are A and B. (Hint: If N is a proper normal subgroup different from A and B, consider the commutator $[N, A]$.)

(b) Give an example to show that (a) would be false were A and B allowed to be abelian.

3. Let $G = \mathrm{GL}_2(\mathbb{R})$ act on $\mathbb{R}^2$ by matrix multiplication: $A \cdot \mathbf{v} = A\mathbf{v}$ for $A \in G$, $\mathbf{v} \in \mathbb{R}^2$.

(a) Determine the stabilizer in G of $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$.

(b) Determine the orbit of $\mathbf{e}_2$.

(c) How many orbits are there for the action of G on $\mathbb{R}^2$?

4. Let G be a group.

(a) Define the ascending central series

$$Z_0(G) \leq Z_1(G) \leq Z_2(G) \leq \cdots$$

for G. (This is also known as the upper central series.)

(b) Prove that the groups $Z_i(G)$ are all characteristic subgroups of G.

(c) Give an example of a nontrivial group G such that the groups $Z_i(G)$ are all trivial.

5. Let $n \geq 2$ and let Z_n be a cyclic group of order n. Prove that

$$\mathrm{Aut}(Z_n) \cong (\mathbb{Z}/n\mathbb{Z})^\times.$$