## Second Semester Algebra Exam

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let $V$ be a finite-dimensional vector space over a field $F$ and $T \in$ $\operatorname{End}_{F}(V)$ a linear transformation. Prove that there is a unique monic polynomial $m(x)$ in $F[x]$ with the property that $m(x)$ has the minimum degree among nonzero polynomials $f(x)$ such that $f(T)$ is the zero transformation.
2. Let $n \geq 1$ and let $K=\mathbb{Q}(\sqrt{1}, \sqrt{2}, \ldots, \sqrt{n})$. Prove that $2^{1 / 3} \notin K$.
3. Let $R$ be a commutative ring with $1 \neq 0$. Prove that $R$ has a minimal prime ideal. (In other words, show that $R$ contains a prime ideal $P$ such that if $Q \subseteq P$ is a prime ideal of $R$ then $Q=P$.) You may assume that $R$ contains a maximal ideal.
4. Prove that the ring $\mathbb{Z}[\sqrt{-2}]=\{a+b \sqrt{-2}: a, b \in \mathbb{Z}\}$ is a Euclidean domain with respect to the norm $\mathrm{N}(a+b \sqrt{-2})=a^{2}+2 b^{2}$.
5. Say that an $R$-module $M$ is torsion if for every $x \in M$ there is $r \in R$ such that $r \neq 0$ and $r x=0$. The annihilator of $M$ is defined to be

$$
\operatorname{Ann}(M)=\{s \in R: s x=0 \text { for all } x \in M\} .
$$

Prove that if $M$ is a finitely generated torsion module over an integral domain $R$ then $\operatorname{Ann}(M) \neq\{0\}$. Give an example of a torsion $R$ module $M$ over an integral domain $R$ such that $\operatorname{Ann}(M)=\{0\}$.

