First-Semester Algebra Exam

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. State and prove Cayley's Theorem about finite groups.

2. Let G be a group of order p^n, where p is prime and $n \geq 1$.

   (a) Prove that G has nontrivial center.

   (b) Using (a), or otherwise, prove that if H is a proper subgroup of G, then H is proper in $N_G(H)$.

3. (a) Let G be a group. Prove that G cannot be expressed as the union of two proper subgroups. (Do not assume that G is finite.)

   (b) Give an example of a group G such that G is the union of three of its proper subgroups.

   (c) Determine all groups G such that G is the union of all its proper subgroups.

4. Prove that there is no simple group of order $9477 = 3^6 \cdot 13$. (Hint: Consider the action of G by conjugation on the set $\mathrm{Syl}_3(G)$ of Sylow 3-subgroups of G.)

5. Classify the groups of order $57 = 19 \cdot 3$ up to isomorphism.