First-Semester Algebra Exam

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

- 1. Let G be a group of order 44. Prove that
 - (a) a Sylow 11-subgroup of G is normal in G, and
 - (b) the centre of G contains an element of order 2.
- 2. Let $G = A \times B$ be a finite group which is the (internal) direct product of two subgroups A and B which are nonabelian simple groups.
 - (a) Show that the only proper, nontrivial normal subgroups of G are A and B. (Hint: If N is a proper normal subgroup different from A and B, consider the commutator [N, A].)
 - (b) Give an example to show that (a) would be false were A and B allowed to be abelian.
- 3. Let $G = \operatorname{GL}_2(\mathbb{R})$ act on \mathbb{R}^2 by matrix multiplication: $A \cdot \mathbf{v} = A\mathbf{v}$ for $A \in G, \mathbf{v} \in \mathbb{R}^2$.

(a) Determine the stabilizer in G of
$$\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$$
.

- (b) Determine the orbit of \mathbf{e}_2 .
- (c) How many orbits are there for the action of G on \mathbb{R}^2 ?
- 4. Let G be a group.
 - (a) Define the ascending central series

$$Z_0(G) \le Z_1(G) \le Z_2(G) \le \dots$$

for G. (This is also known as the upper central series.)

- (b) Prove that the groups $Z_i(G)$ are all characteristic subgroups of G.
- (c) Give an example of a nontrivial group G such that the groups $Z_i(G)$ are all trivial.

5. Let $n \geq 2$ and let Z_n be a cyclic group of order n. Prove that $\operatorname{Aut}(Z_n) \cong (\mathbb{Z}/n\mathbb{Z})^{\times}$.